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ASYMPTOTIC SOLUTIONS OF CONTACT PROBLEMS OF ELASTICITY THEORY FOR MEDIA INHOMOGENEOUS IN DEPTH*

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There are considered the dual integral equations generated by contact problems for half-spaces and half-planes inhomogeneous with depth. There is extended the method from /1/ for the construction of asymptotic solutions of the problems under consideration. Correctness of classes and the solvability of equations are established, and the approximate method proposed for their solution is given a foundation.

1. Contact problems for half-spaces and half-planes inhomogeneous with depth reduce, in a number of cases /2-4/, to finding the solution of a dual integral equation of the form (λ is a geometric parameter)

$$\int_c^d T(\alpha)\rho(\alpha)L(\alpha\lambda)B(\alpha,x)\,d\alpha = f(x), \quad |x|\leqslant 1 \tag{1.1}$$
$$\int_c^d T(\alpha)B(\alpha,x)\,d\alpha = 0, \quad |x|>1$$

In particular, in the problem of shear of an inhomogeneous half-space by a stamp /2/ (problem 1) and in the problem of impression of a stamp in an inhomogneous half-plane (problem 2)

$$\rho(\alpha) = |\alpha|^{-1}, \quad B(\alpha,x) = \bar{e}^{i\alpha x}, \quad c = -d = \infty$$

where (1.1) is considered with the additional condition

$$\int_{-1}^{1}\tau(x)\,dx = P, \quad \tau(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}T(\alpha)e^{-i\alpha x}\,d\alpha, \quad \int_{-1}^{1}\tau(\xi)e^{i\alpha\xi}\,d\xi = T(\alpha) \tag{1.2}$$

and P is the shearing force (problem 1) or the impressing force (problem 2) acting on unit length of stamp. Find $\tau(x)$.

For problems on the torsion by a circular stamp /3/ (problem 3) and on the impression of a circular stamp in an elastic half-space that is inhomogeneous with depth /4/ (problem 4)

$$\rho(\alpha) = \alpha^{-1}, \quad B(\alpha,x) = \alpha J_k(\alpha x), \quad c = 0, \quad d = \infty$$

$k = 1$ for problem 3, and $k = 0$ for problem 4. Find $\tau(x)$. Here

$$\tau(x) = \int_0^\infty T(\alpha)\alpha J_k(\alpha x)\,d\alpha, \quad T(\alpha) = \int_0^1 \tau(\rho)J_k(\alpha\rho)\rho\,d\rho$$

Upon satisfying the conditions

$$\min_{y\in(0,\infty)} G(y)\geqslant c_1>0, \quad \max_{y\in(0,\infty)} G(y)\leqslant c<\infty$$
$$\lim_{y\to\infty} G(y) = \text{const} \quad \text{(problems 1 and 3)}$$

$$\min_{y\in(0,\infty)} \theta(y)\geqslant c_1>0, \quad \max_{y\in(0,\infty)} \theta(y)\leqslant c<\infty$$
$$\theta(y) = \frac{2\mu(y)[\lambda(y)+\mu(y)]}{[\lambda(y)+2\mu(y)]}$$
$$\lim_{y\to\infty}\theta(y) = \text{const} \quad \text{(problems 2 and 4)}$$

where $G(y)$ is the shear modulus and $\mu(y)$ and $\lambda(y)$ are Lamé coefficients of the half-space, and y is the distance from the surface of the medium, it can be shown /2-4/ that the transforms of the kernel $L(u)$ possess the following properties (B and D are constants):

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$$L(u) = A + B\,|u| + O(u^2), \quad u \to 0 \tag{1.3}$$

$$L(u) = 1 + D\,|u|^{-1} + O(u^{-2}), \quad u \to \infty \tag{1.4}$$

$$A = \lim_{y\to\infty} G(0)\,G^{-1}(y) \ \text{(problems 1,3)} \quad A = \lim_{y\to\infty} \theta(0)\,\theta^{-1}(y) \ \text{(problems 2,4)} \tag{1.5}$$

For multilayer media the properties of the compliance functions, analogous to (1.5), were noted in /5/. The properties (1.5) mean that the value $L(0)$ for the problems under consideration is independent of the manner in which the elastic moduli vary in the half-space from $y = 0$ to $y \to \infty$, and are determined only by their values for $y = 0$ and $y \to \infty$. Graphically this appears as follows: if the set of curves describing certain laws of variation of the elastic moduli with depth have identical values on the surface of the half-space and as $y \to \infty$, then the graphs of the corresponding transforms $L(u)$ of problems 1 and 3 will issue from a common point $L(0) = A$ and converge at one point $L(\infty) = 1$.

Let us introduce the following definitions:

Definition 1.1. The function $L(u)$ belongs the class Π_N if $L(\alpha\Lambda)$ has the form

$$L(\alpha\lambda) = L_N(\alpha\lambda) = \prod_{i=1}^{N} (\alpha^2 + A_i^2\lambda^{-2})(\alpha^2 + B_i^2\lambda^{-2})^{-1} \tag{1.6}$$

$$(B_i - B_k)_{i\neq k}(A_i - A_k) \neq 0$$

Here A_i, B_i $(i = 1, 2, \ldots, N)$ are certain constants.

Definition 1.2. The function $L(u)$ belongs to the class Σ_M if $L(\alpha\lambda)$ has the form

$$L(\alpha\lambda) = L_M^{\Sigma}(\alpha\lambda) \equiv \sum_{k=1}^{M} \frac{c_k\lambda^{-1}|\alpha|}{\alpha^2 + D_k^2\lambda^{-2}}$$

Definition 1.3. The function $L(u)$ belongs to the class $S_{N,M}$ if it has the form

$$L(\alpha\lambda) = L_N(\alpha\lambda) + L_M^{\Sigma}(\alpha\lambda) \tag{1.7}$$

We show that expressions of the form (1.7) can be approximated by $L(u)$ with the properties (1.3) and (1.4). To do this, we use the lemma /6,7/.

Lemma 1.1. Let an even real function $\varphi(u)$ continuous on the whole real axis vanish at infinity, then it allows approximation in $C(-\infty, \infty)$ by series of functions of the form

$$\varphi_k = (u^2 + D_k^2)^{-1}$$

We apply this lemma to prove the following assertion.

Theorem 1.1. Under conditions that the function $L(u)$ possesses the properties (1.3) and (1.4), it allows approximation by expressions of the form (1.7).

Proof. We select the constants A_i and B_i $(i = 1, 2, \ldots N)$ in (1.6) such that

$$\prod_{i=1}^{N} (A_i^2 B_i^{-2}) = A \tag{1.8}$$

We consider the function

$$L_\Sigma(u) = (L(u) - L_N(u))\,|u|^{-1} \tag{1.9}$$

On the basis of the properties (1.3) and (1.4) and condition (1.8), it follows that $L_\Sigma(u)$ satisfies the condition of the Lemma 1.1. This means that the following representation holds

$$L_\Sigma(u) = \sum_{k=1}^{\infty} c_k (u^2 + D_k^2)^{-1} \tag{1.10}$$

Or from conditions (1.9) and (1.10)

$$L(u) = L_N(u) + |u| \sum_{k=1}^{\infty} c_k (u^2 + D_k^2)^{-1}$$

2. Let us consider the bilateral asymptotic solution for small and large values of λ ($\lambda \to 0$, $\lambda \to \infty$) for problems 1 and 2.

Existence and uniqueness of the solution of the integral equation in problems 1 and 2 for $L(u)$ of class Π_N. Let $W_p^k(a, b)$ be a Sobolev space of functions for which all possible generalized derivatives for order k inclusive are summable in the segment $[a, b]$ with power p /8/. Let $B_k^\alpha(a, b)$ be a space of functions having all derivatives to order k inclusive on the segment $[a, b]$, whose k-th order derivatives satisfy the Hölder condition with index α, with the usual norm /9/:

$$\|f\|_{B_k^\alpha(a,b)} = \sum_{i=0}^{k} \max_{x\in[a,b]} |f^{(i)}(x)| + \max_{x,y\in[a,b]} |f^{(k)}(x) - f^{(k)}(y)|\,|x-y|^{-\alpha}$$

Let $C_\gamma^{(k)}(a, b)$ denote the space of functions whose k-th derivatives are continuous with weight $(x-a)^\gamma(b-x)^\gamma$ with the norm /9/:

$$\|f\|_{C_\gamma^{(k)}(a,b)} = \sum_{i=0}^{k-1} \max_{x\in[a,b]} |f^{(i)}(x)(x-a)^{\gamma-k+i}(b-x)^{\gamma-k+i}|$$

We denote the subspace $C_\gamma^{(k)}(a, b)$ of even functions by $C_\gamma^{(k)+}(a, b)$. We denote the subspace $C_\gamma^{(k)}(a, b)$ of odd functions by $C_\gamma^{(k)-}(a, b)$.

We use the following lemma below.

Lemma 2.1. /10/. Let the function $f(x)$ correspond on the segment (-1,1) to the Fourier series $a_1\cos\pi x + a_2\cos 2\pi x + \ldots$, then the series $|a_1| + |2a_2| + \ldots$ converges if $f(x)\in B_1^{1/2+\varepsilon}(-1, 1)$, $\varepsilon > 0$.

Lemma 2.2. Equation (1.1) with the additional condition (1.2) for problems 1 and 2 is solvable uniquely for $L(u)$ of class Π_N. If $f(x)$ is an even function and belongs to $B_1^{1/2+\varepsilon}(-1, 1)$, $\varepsilon > 0$; hence, the estimate

$$\|\tau(x)\|_{C_{1/2}^{(0)+}(-1,1)} \leqslant m(\Pi_N)\,\|f\|_{B_1^{1/2+\varepsilon}(-1,1)}, \quad m(\Pi_N) = \mathrm{const} \tag{2.1}$$

holds in the class of functions $C_{1/2}^{(0)+}(-1, 1)$

Below, $m(A)$ shall denote a certain constant dependent on the specific form of the functions belonging to the class A.

Proof. We represent the right side of the first equation in (1.1) as a Fourier series without limiting the generality, we consider ($f(x)$ is an even function)

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos k\pi x$$

This can always be done under the conditions of Lemma 2.2.

Using the method shown in /1/, we obtain an expression for the stress /2/

$$\tau(x) = \frac{P}{\pi\sqrt{1-x^2}} + \sum_{i=1}^{N} C_i \Phi\left(\frac{A_i}{\lambda}; x\right) - \sum_{k=1}^{\infty} \frac{a_k k\pi}{L_N(k\pi)} F(k\pi, x) \tag{2.2}$$

$$\Phi(A, x) = -\frac{I_1(A)}{\sqrt{1-x^2}} + A\int_x^1 [I_0(A)\alpha\,\mathrm{ch}\,A(\alpha-x) - I_1(A)\,\mathrm{ch}\,A(\alpha-x)]\frac{d\alpha}{\sqrt{1-\alpha^2}}$$

$$F(l, x) = -\frac{J_1(l)}{\sqrt{1-x^2}} + l\int_x^1 [J_0(l)\alpha\cos l(\alpha-x) - J_1(l)\sin l(\alpha-x)]\frac{d\alpha}{\sqrt{1-\alpha^2}}$$

The constants C_i are determined from the system of linear algebraic equations

$$\sum_{i=1}^{N} C_i S\left(\frac{A_i}{\lambda}; \frac{B_k}{\lambda}\right) + \frac{PK_0\left(\frac{B_k}{\lambda}\right)}{\frac{B_k}{\lambda}} + \sum_{m=1}^{\infty} \frac{a_m m\pi}{h_N(m\pi)} Z\left(m\pi; \frac{B_k}{\lambda}\right) = 0, \quad k = 1, 2, \ldots N \tag{2.3}$$

$$S(A; B) = [AI_0(A)K_1(B) + BI_1(A)K_0(B)](A^2 - B^2)^{-1}$$
$$Z(m; B) = [mJ_0(m)K_1(B) + BJ_1(m)K_0(B)](B^2 + m^2)^{-1}$$

The system (2.3) is evidently solvable if A_i, B_i satisfy conditions (1.6).

Let us estimate the expression in the right side of (2.2). Under the condition of convergence of the series therein, (2.2) and (2.3) have meaning. Using the asymptotic properties of the Bessel functions /11/, and the asymptotic estimates of incomplete cylindrical functions in the Poisson form /12/, we obtain that the series in (2.2) and (2.3) converge if the following series converge:

$$\Sigma_1^{(1)} = \sum_{k=1}^{\infty} a_k k^{1/2}, \quad \Sigma_1^{(2)} = \sum_{k=1}^{\infty} a_k k \sin k\pi x, \quad \Sigma_1^{(3)} = \sum_{k=1}^{\infty} a_k k \cos k\pi x \tag{2.4}$$

On the basis of Lemma 2.1 we conclude that series of the form (2.4) are convergent, hence an estimate of the form (2.1) follows and so does the uniqueness of the solution constructed.

There holds the more general lemma:

Lemma 2.3. Equation (1.1) with the additional condition (1.2) for problems 1 and 2 is solvable uniquely for $L(u)$ of class Π_N if $f(x)$ is an even function and belongs to $B_{k+1}^{1/2+\varepsilon}(-1, 1)$, $\varepsilon > 0$, the following estimate hence holds in the class of functions $C_{k+1/2}^{(k)+}(-1, 1)$

$$\| \tau(x) \|_{C_{k+1/2}^{(k)}(-1,1)} \leqslant m(\Pi_N, k) \| f \|_{B_{k+1}^{1/2+\varepsilon}(-1,1)}$$

Lemma 2.3 is proved analogously to Lemma 2.2 by using estimates of the right sides of (2.3).

We shall also denote the integral operator corresponding to the function $L(u)$ belonging to class A by A below.

We write equation (1.1) for $L(u)$ of class Π_N in terms of operators as

$$\Pi_N \tau = f \tag{2.5}$$

Theorem 2.1. (Corollary of Lemma 2.3). If the conditions of Lemma 2.3 are satisfied, then the following estimate holds

$$\| \tau(x) \|_{C_{k+1/2}^{(k)}(-1,1)} \leqslant \| \Pi_N^{-1} \| m(k) \| f \|_{B_{k+1}^{1/2+\varepsilon}(-1,1)}$$

Existence and uniqueness of the solution of the integral equation of problems *1* and *2* for $L(u)$ of the class $S_{N,M}$. Equation (1.1) can be written in terms of the operator for $L(u)$ of class $S_{N,M}$ in the form

$$\Pi_N \tau + \Sigma_M \tau = f \tag{2.6}$$

Lemma 2.4. The operator $\Pi_N^{-1}\Sigma_M$ of problems 1 and 2 is a compression operator in the space $C_{k+1/2}^{(k)+}(-1, 1)$ upon compliance with the conditions in Lemma 2.3 imposed on $f(x)$ if $0 < \lambda < \lambda^*$ or $\lambda > \lambda^0$, where λ^* and λ^0 are certain fixed values of λ.

Proof. We prove the lemma for $k = 0$. For $k > 0$ the proof is analogous. Let us consider the operator $\Sigma_M(\tau)$. Without limiting the generality, we set $M = 1$. We have

$$\Sigma_1(\tau) = \frac{c\pi}{D} \int_{-1}^{1} \tau(\xi) \exp\left[-\frac{D}{\lambda}(\xi - x)\right] d\xi$$

We represent $\Sigma_1(\tau)$ in the form of the series

$$\Sigma_1(\tau) = \frac{c_0}{2} + \sum_{k=1}^{\infty} c_k \cos k\pi x$$

We find the coefficients c_k from the following formulas

$$c_k = \frac{4\pi c \lambda^{-1}}{D^2\lambda^{-2} + (k\pi)^2} \left[(-1)^{k+1} \exp\left(-\frac{D}{\lambda}\right) \int_0^1 \tau(\xi) \operatorname{ch} \frac{D}{\lambda} \xi d\xi + \int_0^1 \tau(\xi) \cos k\pi\xi \, d\xi\right], \quad k = 0, 1, 2, \ldots \tag{2.7}$$

Utilizing (2.7), we find the following estimates

$$\max_{x \in (-1,1)} | \Sigma_1(\tau) \sqrt{1 - x^2} | \leqslant \sum_{k=0}^{\infty} | c_k | \leqslant \lambda M^*, \quad \lambda \to 0 \quad (\lambda < \lambda^*)$$

$$\max_{x \in (-1,1)} | \Sigma_1(\tau) \sqrt{1 - x^2} | \leqslant \sum_{k=0}^{\infty} | c_k | \leqslant \lambda^{-1} M^0, \quad \lambda \to \infty \quad (\lambda > \lambda^0)$$

where the constants M^* and M^0 are independent of λ. Hence, by using estimates analogous to the estimates in Lemmas 2.2 and 2.3, we obtain that λ can be selected in such a manner that the operator $\Pi_N^{-1}\Sigma_M$ will be a compression operator /13/ under the condition of this lemma.

On the basis of Lemma 2.4 by applying the Banach principle of compressed mappings to the equation

$$\tau+\Pi_N^{-1}\Sigma_M\tau=\Pi_N^{-1}f \tag{2.8}$$

we obtain the proof of the existence and uniqueness of the solution of (2.6) under the constraints imposed.

There is therefore proved the following theorem.

Theorem 2.2. Equation (1.1) with the additional condition (1.2) (problems 1 and 2) is uniquely solvable in the space $C^{(k)+}_{k+1/2}(-1,1)$ for $L(u)$ of class $S_{N,M}$ if $f(x)$ is an even function and belongs to $B^{1/2+\varepsilon}_{k+1}(-1,1)$, $\varepsilon>0$ for $0<\lambda<\lambda^*$ or $\lambda>\lambda^\circ$ where λ^* and λ° are certain fixed values of λ and the following estimate holds

$$\|\tau(x)\|_{C^{(k)+}_{k+1/2}(-1,1)}\leqslant m(\Pi_N,S_M,k)\|f\|_{B^{1/2+\varepsilon}_{k+1}(-1,1)}$$

Finally, we formulate the following theorem.

Theorem 2.3. Equation (1.1) with the additional condition (1.2) is solvable uniquely in the space $C^{(k)+}_{k+1/2}(-1,1)$ for problems 1 and 2 if $f(x)$ is even function and belongs to $B^{1/2+\varepsilon}_{k+1}(-1,1)$, $\varepsilon>0$ for $0<\lambda<\lambda^*$ or $\lambda>\lambda^\circ$, where λ^* and λ° are certain fixed values of λ and the following estimate holds

$$\|\tau(x)\|_{C^{(k)+}_{k+1/2}(-1,1)}\leqslant m(\Pi_N,\Sigma_\infty,k)\|f\|_{B^{1/2+\varepsilon}_{k+1}(-1,1)}$$

Theorem 2.3 follows from the assertions of Theorems 1.1 and 2.2 and is proved by using the method known in perturbation theory which is based on successive approximations, exactly as in /9/.

3. We consider the bilateral asymptotic solution of problem 3 for small and large values of λ ($\lambda\to0,\lambda\to\infty$).

Existence and uniqueness of the solution of the integral equation of problem 3 for $L(u)$ of Class Π_N. We assume the results of the theory of Fourier–Bessel series known /14/. Moreover, we utilize the following assertions.

Lemma 3.1. /15/. Let a function $f(x)$ be defined and differentiable $2s$ times in the segment $[0,1]$ $(s>1)$, where

$$1)\ f(0)=f^{(1)}(0)=\ldots f^{(2s-1)}(0)=0;$$

2) $f^{(2s)}(x)$ is bounded (this derivative can also not exist at individual points);

$$3)\ f(1)=\ldots=f^{(2s-2)}(1)=0.$$

Then the inequality

$$|a_n|\leqslant c\lambda^{-(2S+1/2)}\quad(c=\text{const})$$

is valid for the coefficients of the Fourier–Bessel function $f(x)$

Lemma 3.2 /15/. Under the conditions of Lemma 3.1 for $s\geqslant1$

$$p\geqslant0,\quad|a_nJ_p(\lambda_nx)|\leqslant H\lambda_n^{-(2s-1/2)}\quad(H=\text{const}),\quad\forall x\in[0,1]$$
$$p\geqslant-1/2,\quad|a_nJ_p(\lambda_nx)|\leqslant Lx^{-1/2}\lambda_n^{-2S}\quad(L=\text{const}),\quad\forall x\in[0,1]$$

Definition 3.1. We shall say that the function $f(x)$, absolutely integrable on a segment $[0,1]$, satisfies the condition M_k $(k=0,1)$ if a Fourier–Bessel expansion holds ($M_f^k(-1,1)$ is a certain constant)

$$f(x)=\sum_{n=1}^{\infty}a_n^kI_k(\lambda_n^kx),\ \sum_{n=1}^{\infty}|a_n^k\lambda_n^k|\leqslant M_f^k(-1,1)<\infty,\quad k=0,1 \tag{3.1}$$

We note that the conditions of Lemma 3.1 for $s=2$ are sufficient for the inequality (3.1) to hold.

We consider (1.1) for an $f(x)$ of the following form

$$f(x)=G_0\varepsilon\,[x+2\chi(x)] \tag{3.2}$$

Lemma 3.3. If $\chi(x)$ is an odd function and satisfies the condition $\boldsymbol{M_1}$, then (1.1) of problem 3 is uniquely solvable for $L(u)$ of the class Π_N in the class of functions $C^{(0)-}_{1/2}(-1,1)$, and the following estimate holds

$$\|\tau(x)\|_{C^{(0)-}_{1/2}(-1,1)}\leqslant m(\Pi_N)M_\chi^1(1,1) \tag{3.3}$$

Proof. Since the function $\chi(x)$ satisfies the condition M_1, then it can be represented in the form of the Fourier–Bessel series

$$\chi(x)=\sum_{k=1}^{\infty} b_k J_1(\mu_k x)$$

Utilizing the method in /1/, we obtain an expression for the stress /3/

$$\tau(x)=\frac{4G_0(0)\varepsilon}{\pi}\left[L_N^{-1}(0)\frac{x}{\sqrt{1-x^2}}+\sum_{i=1}^{N} C_i z\left(x;\frac{A_i}{\lambda}\right)\right] \tag{3.4}$$

$$S(x;\mu)=\frac{\sin x\mu}{x}+\frac{x\sin\mu}{\sqrt{1-x^2}\,(1+\sqrt{1-x^2})}-\mu x\int_x^1\frac{\cos\mu t\,dt}{\sqrt{t^2-x^2}\,(t^2+\sqrt{t^2-x^2})}$$

$$Z(x;A)=\frac{\sin Ax}{x}+\frac{x\,\mathrm{sh}\,A}{\sqrt{1-x^2}\,(1+\sqrt{1-x^2})}-$$
$$Ax\int_x^1\frac{\mathrm{ch}\,At\,dt}{\sqrt{t^2-x^2}\,(t^2+\sqrt{t^2-x^2})}$$

The constants C_i are determined from the system of linear algebraic equations

$$\sum_{i=1}^{N} C_i p\left(\frac{B_k}{\lambda};\frac{A_i}{\lambda}\right)+\frac{1+B_k\lambda^{-1}}{L_N(0)B_k^2}+\sum_{j=1}^{\infty} b_j d\left(\frac{B_k}{\lambda};\mu_j\right)=0 \quad (k=1,2,\ldots,N) \tag{3.5}$$

$$p(B;A)=\frac{A\,\mathrm{ch}\,A+B\,\mathrm{sh}\,A}{B^2-A^2};\quad d(B;\mu)=\frac{\mu\cos\mu+B\sin\mu}{L_N(\lambda\mu)\,(B^2+\lambda^2\mu^2)}$$

The system (3.5) is solvable uniquely if A_i, B_k satisfy conditions (1.6). Considering (3.4) and (3.5), we see that the assertion of the lemma and the estimate (3.3) hold when the estimate (3.1) (condition M_1) is satisfied. In this case the relation between the moment and the angle of rotation has the form

$$M=16G_0(0)\varepsilon a^3\left[\frac{1}{3}L_N^{-1}(0)+\sum_{i=1}^{N} C_iF(A_i\lambda^{-1})+\sum_{j=1}^{\infty} b_jL_N^{-1}(\lambda\mu_j)R(\mu_j\lambda^{-1})\right]$$

where we have introduced the notation

$$F(A)=A^{-2}(A\,\mathrm{ch}\,A-\mathrm{sh}\,A);\quad R(\mu)=\mu^{-2}(\sin\mu-\mu\cos\mu)$$

We consider (1.1) for problem 3, as written in the form (2.5) (considering $f(x)$ to have the form (3.2)).

Theorem 3.1. (Corollary to Lemma 3.3). If the conditions of Lemma 3.3 hold, then the operator Π_N is reversible and the following estimate holds

$$\|\tau(x)\|_{C_{1/2}^{(0)-}(-1,1)}\leqslant\|\Pi_N^{-1}\|\,mM_x^{1}(-1,1)$$

Existence and uniqueness of the solution of the integral equation in problems 3 for $L(u)$ of the class $S_{N,M}$. We consider equation (1.1) of problem 3 for $f(x)$ in the form (3.2) and $L(u)$ of class $S_{N,M}$ written in the operator form (2.6).

Lemma 3.4. The operator $\Pi_N^{-1}\Sigma_M$ of problem 3 is a compression operator in the space $C_{1/2}^{(0)-}(-1,1)$ upon satisfaction of the conditions in Lemma 3.3 if $0<\lambda<\lambda^*$ or $\lambda>\lambda^\circ$, where λ^* and λ° are certain fixed values of λ.

Proof. We consider the operator $\Sigma_M(\tau)$. Without limiting the generality, we set $M=1$ and we have

$$\Sigma_1(\tau)=\mathrm{Def}\int_0^1\tau(\rho)\rho\left[\int_0^\infty\frac{c\lambda^{-1}\gamma}{\gamma^2+D^2\lambda^{-2}}J_1(\gamma x)J_1(\gamma\rho)\,d\gamma\right]d\rho$$

We represent $\Sigma_1(\tau)$ as the Fourier–Bessel series

$$\Sigma_1(\tau)=\sum_{k=1}^{\infty} a_kJ_1(\mu_kx)$$

The coefficients a_k are found from the following formulas

$$a_k = \frac{2c\lambda^{-1}}{I_2{}^2(\mu_k)(\mu_k{}^2 + D^2\lambda^{-2})}\left[\int_0^1 \tau(\rho)\rho J_1(\rho\mu_k)\,d\rho - \right. \tag{3.6}$$

$$\left. \mu_k J_0(\mu_k) K_1\left(\frac{D}{\lambda}\right) \int_0^1 \tau(\rho)\rho J_1\left(\frac{\rho D}{\lambda}\right) d\rho\right]$$

Using the asymptotic estimates of cylindrical functions of imaginary argument /11/, we obtain the following estimates from (3.6):

$$\max_{x\in(-1,1)} |\Sigma_1(\tau)\sqrt{1-x^2}| \leqslant C\sum_{k=1}^{\infty} |a_k| \leqslant \lambda M^*, \quad \lambda\to 0 \quad (\lambda<\lambda^*)$$

$$\max_{x\in(-1,1)} |\Sigma_1(\tau)\sqrt{1-x^2}| \leqslant C\sum_{k=1}^{\infty} |a_k| \leqslant \lambda^{-1} M^0, \quad \lambda\to \infty \quad (\lambda>\lambda^\circ)$$

where the constants M^* and M° are independent of λ. Hence, analogously to the estimates in Lemma 3.3 we obtain that λ can be selected in such a manner that the operator $\Pi_N{}^{-1}\Sigma_M$ will be a compression operator /13/ under the conditions of this lemma. On this basis, by applying the Banach principle of compressed mappings to an equation of the form (2.8), we obtain the proof of the existence and uniqueness of the solution of (1.1) under the constraints imposed. This means that the following estimates hold.

Theorem 3.2. Equation (1.1) of problem 3 is solvable uniquely in the space $C_{1/2}^{(0)-}(-1, 1)$ for $L(u)$ of the class $S_{N,M}$ if $\chi(x)$ is an odd function and satisfies the condition M_1 for $0<\lambda<\lambda^*$ or $\lambda>\lambda^\circ$ and the following estimate holds:

$$\|\tau(x)\|_{C_{1/2}^{(0)-}(-1,1)} \leqslant m(\Pi_N, \Sigma_M) M_\chi{}^1(-1, 1)$$

Finally, we formulate the following theorem.

Theorem 3.3. Equation (1.1) is solvable uniquely for problem 3 in the space $C_{1/2}^{(0)-}(-1, 1)$ if $\chi(x)$ is an odd function and satisfies condition M_1 for $0<\lambda<\lambda^*$ or $\lambda>\lambda^\circ$, where λ^* and λ° are certain fixed values of λ, hence, the following estimate holds:

$$\|\tau(x)\|_{C_{1/2}^{(0)-}(-1,1)} \leqslant m(\Pi_N, \Sigma_\infty) M_\chi{}^1(-1, 1)$$

The proof of Theorem 3.3 follows from the assertions in Theorems 1.1 and 3.2 and is analogous to that carried out in /9,16/.

4. We consider the bilateral asymptotic solution of problem 4 for small and large values of λ $(\lambda\to 0, \lambda\to\infty)$.

Existence and uniqueness of the solution of the integral equation of problem *4* for $L(u)$ of the class Π_N. Let us consider equation (1.1) of problem 4 for an $f(x)$ of the following kind:

$$f(x) = \theta_0(0)\,\delta\,[1 + \varphi(x)] \tag{4.1}$$

Lemma 4.1. If $\varphi(x)$ is an even function and satisfies the condition M_0, then (1.1) for problem 4 is solvable uniquely in the class of functions $C_{1/2}^{(0)+}(-1, 1)$ for $L(u)$ of the class Π_N and the following estimate holds:

$$\|\tau(x)\|_{C_{1/2}^{(0)+}(-1,1)} \leqslant m(\Pi_N) M_\varphi{}^0(-1, 1) \tag{4.2}$$

Proof. Since $\varphi(x)$ satisfies the condition M_0, then it can be represented in the form of a Fourier–Bessel series

$$\varphi(x) = \sum_{k=1}^{\infty} b_k J_0(\mu_k x)$$

Utilizing the method in /1/, we obtain an expression for the stress /4/

$$\tau(x) = 2\theta_0(0)\,\delta\pi^{-1}\left[L_N^{-1}(0)\frac{1}{\sqrt{1-x^2}} + \sum_{i=1}^{N} C_i\psi\left(x, \frac{A_i}{\lambda}\right) + \right. \tag{4.3}$$

$$\left. \sum_{j=1}^{\infty} b_j L_N^{-1}(\lambda\mu_j) f(x; \mu_j)\right]$$

Here

$$\psi(x, A)=\frac{\operatorname{ch} A}{\sqrt{1-x^2}}-A\int_x^1\frac{\operatorname{sh} At\,dt}{\sqrt{t^2-x^2}};$$

$$f(x, \varepsilon)=\frac{\cos\varepsilon}{\sqrt{1-x^2}}+\varepsilon\int_x^1\frac{\sin t\varepsilon\,dt}{\sqrt{t^2-x^2}}$$

The constants C_i are determined from the system of linear equations

$$\sum_{i=1}^{\infty} C_i\alpha\left(\frac{B_k}{\lambda};\frac{A_i}{\lambda}\right)+L_N^{-1}(0)\lambda B_k^{-1}+ \qquad (4.4)$$

$$\sum_{j=1}^{\infty} b_j\beta\left(\frac{B_k}{\lambda};\mu_j\right)=0,\quad k=1,2,\ldots,N$$

where

$$\alpha(B, A)=\frac{B\operatorname{ch} A+A\operatorname{sh} A}{B^2-A^2};\quad \beta(B,\mu)=\frac{B\cos\mu-\mu\sin\mu}{L_N(\lambda\mu)(B^2+\lambda^2\mu^2)}$$

The system (4.4) is solvable uniquely if A_i, B_k satisfy conditions (1.6). Considering (4.3) and (4.4), it can be seen that if the function $\varphi(x)$ in the right side of (4.1) satisfies the condition M_0, then the series in (4.3) and (4.4) are convergent, hence the assertion of the lemma and the estimate (4.2) follow. In this case the relation between the applied force and the settling of the stamp has the form

$$P=4\pi a\theta_0(0)\left[L_N^{-1}(0)+\sum_{i=1}^{N} C_iA_i^{-1}\lambda\operatorname{sh} A_i\lambda^{-1}+\sum_{j=1}^{\infty} b_jL_N^{-1}(\lambda\mu_j)\,\mu_j^{-1}\lambda\sin\mu_j\lambda^{-1}\right]$$

We consider equation (1.1) for problem 4 written in the form (2.5) (considering $f(x)$ to have the form (4.1)).

Theorem 4.1. (corollary of Lemma 4.1). If the condition of Lemma 4.1 hold, then the operator Π_N is reversible and the following estimate holds:

$$\|\tau(x)\|_{C_{1/2}^{(0)+}(-1,1)}\leqslant\|\Pi_N^{-1}\|\,mM_\varphi{}^\circ(-1,1)$$

Existence and uniqueness of the solution of the integral equation of problem 4 for $L(u)$ of class $S_{N,M}$ We consider equation (1.1) of problem 4 for an $f(x)$ of the form (4.1) and $L(u)$ of the class $S_{N,M}$ written in the operator form (2.6).

Lemma 4.2. The operator $\Pi_N^{-1}\Sigma_M$ of problem 4 is a compression operator in the space $C_{1/2}^{(0)+}(-1,1)$ upon satisfaction of the conditions of Lemma 4.1 if $0<\lambda<\lambda^*$ or $\lambda>\lambda^0$, where λ^* and λ^0 are certain fixed values of λ. The proof is analogous to the proof of Lemma 3.4. This means that the following holds:

Theorem 4.2. Equation (1.1) for problem 4 is solvable uniquely in the space $C_{1/2}^{(0)+}(-1,1)$ for $L(u)$ of the class $S_{N,M}$ if $\varphi(x)$ is an even function and satisfies the condition M_0 for $0<\lambda<\lambda^*$ or $\lambda>\lambda^0$, where λ^* and λ^0 are certain fixed values of λ and the estimate holds

$$\|\tau(x)\|_{C_{1/2}^{(0)}}+(-1,1)\leqslant m(\Pi_N, S_m)M_\varphi{}^\circ(-1,1)$$

Finally, we formulate the following theorem.

Theorem 4.3. Equation (1.1) for problem 4 is solvable uniquely in the space $C_{1/2}^{(0)+}(-1,1)$, if $\varphi(x)$ is an even function and satisfies the condition M_0 for $0<\lambda<\lambda^*$ or $\lambda>\lambda^0$, where λ^* and λ^0 are certain fixed values of λ, and the following estimate holds:

$$\|\tau(x)\|_{C_{1/2}^{(0)+}(-1,1)}\leqslant m(\Pi_N,\Sigma_\infty)M_\varphi{}^\circ(-1,1)$$

The proof of Theorem 4.3 follows from the assertions of Theorems 1.1 and 4.2 and is analogous to that carried out in /9,16/.

For a numerical realization the expansion of the range of applicability of the method elucidated in λ can be achieved because of improvement of the approximation of $L(u)$ by functions of the class Π_N. A good approximation is successfully achieved here by using the following algorithm.

We map the function $L(u)$ by the mapping $\gamma=\frac{u^2}{u^2+c^2}$ from the interval $(0,\infty)$ onto the segment

$(0, 1)$, $(u = c\sqrt{\gamma(\gamma-1)^{-1}})$. We approximate the function $\sqrt{L(\gamma)}$ and $\sqrt{L^{-1}(\gamma)}$ on the segment (0.1) by N-th order Bernshtein polynomials (or by Chebyshev nodes), and we obtain

$$\sqrt{L_N(\gamma)} = \sum_{i=1}^{N} a_i\gamma^i, \quad \sqrt{L_N^{-1}(\gamma)} = \sum_{i=1}^{N} b_i\gamma^i$$

Then

$$L_N(u) = \Big(\sum_{i=1}^{N} a_i{*}u^{2i}\Big)\Big(\sum_{i=1}^{N} b_i u^{2i}\Big)^{-1} \tag{4.5}$$

By determining the roots of the numerator and denominator in (4.5), we find the desired values of A_i, B_i $(i = 1, 2, \ldots, N)$. Such a modification of the method described in /16/ permits avoiding the presence of an N-tiple root in the denominator of the approximation found.

Specific examples of the construction of solutions by the method elucidated were considered the papers /2–4/.

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REFERENCES

1. ALEKSANDROV V.M., On the solution of a class of dual equations. Dokl. Akad. Nauk SSSR, Vol. 210, No.1, 1973.
2. AIZIKOVICH S.M., Shear of an elastic inhomogeneous half-space of special form by a stamp. Izv. Akad. Nauk SSSR, Mekhan. Tverd. Tela, No.5, 1978.
3. AIZIKOVICH S.M., Torsion of an inhomogeneous half-space by a circular stamp. In. Analysis of Plates and Shells. Rostov Struct.-Eng'g Inst., 1978.
4. AIZIKOVICH S.M., Asymptotic methods in the problem of impressing a stamp of circular planform in an inhomogeneous half-space in depth. In. Abstracts of Reports. All-Union Conf. on Elasticity Theory. Izd-vo Akad. Nauk, ArmSSR, Erevan, 1979.
5. PRIVARNIKOV A.K., Spatial deformation of a multilayer foundation. In.Stability and Strength of Structure Elements. Izd-vo Dnepropetrovsk. Univ., 1973.
6. AKHIEZER N.I., Lectures on the Theory of Approximation. NAUKA, Moscow, 1965.
7. BABESHKO V.A., Asymptotic properties of solutions of certain two-dimensional integral equations, Dokl. Akad. Nauk SSSR, Vol.206, No.5, 1972.
8. MIKHLIN S.G., Linear Partial Differential Equaitons. Vysshaia Shkola, Moscow, 1977.
9. BABESHKO V.A., Integral equations of convolution of the first kind on a system of segments occurring in the theory of elasticity and mathematical physics, PMM, Vol.35, No.1, 1971.
10. ZYGMUND A., Trigonometric Series. Vol.1, and Vol.2, MIR, Moscow, 1965. Also Vol.1, Warsaw, 1935.
11. GRADSHTEIN I.S. and RYZHIK I.M., Tables of Integrals, Sums, Series, and Products, NAUKA, Moscow, 1965.
12. AGREST M.M. and M.Z. Maksimov, Theory of Incomplete Cylinder Functions and Their Application, Atomizdat, Moscow, 1965.
13. KANTOROVICH L.V. and AKILOV G.P., Functional Analysis, NAUKA, Moscow, 1977. (See also in English, Functional Analysis in Normed Spaces, Pergamon Press, Book No. 10091, 1964.
14. WATSON G.N., A Treatise on the Theory of Bessel Functions, 2nd Ed. Cambridge University Press, Cambridge, 1948.
15. TOLSTOV G.P., Fourier Series, Fizmatgiz, Moscow, 1960.
16. BABESHKO V.A., New effective method of solving dynamic contact problems, Dokl. Akad. Nauk SSSR, Vol.217, No.4, 1974.

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